Robbins-Monro averaging lemma

We write (z_k) for the whole sequence $z_0, z_1, z_2, ...$; the symbol z_k without the surrounding parentheses denotes the value at index k.

Robbins-Monro averaging lemma. Consider an algorithm that repeatedly updates a single scalar. Before update $k+1$, the algorithm has already determined three numbers: its current value z_k, an averaging weight β_k, and a noise-free input c_k. It then observes a new random perturbation ξ_{k+1} and performs the update

$$z_{k+1} = (1-\beta_k)z_k + \beta_k(c_k + \xi_{k+1}). \tag{R.1}$$

Thus β_k determines how much of the old value is replaced, c_k is the value that would be averaged in without randomness, and ξ_{k+1} is the random error in the newly observed input.

Let H_k denote the complete record of everything the algorithm has observed and computed before ξ_{k+1} is observed. In particular, z_k, β_k, and c_k are fixed once H_k is fixed, whereas ξ_{k+1} has not yet been observed. Assume that (ξ_{k+1}) is a martingale difference sequence with respect to these records and that it is bounded. Thus, for every k,

$$\mathbb{E}[\xi_{k+1} \mid H_k] = 0, \tag{R.2}$$

and there is a constant C such that $|\xi_{k+1}| \le C$.

Suppose $0 \le \beta_k \le 1$ and

$$\sum_k \beta_k = \infty, \qquad \sum_k \beta_k^2 < \infty. \tag{R.3}$$

Suppose also that

$$\limsup_k |c_k| \le A. \tag{R.4}$$

Then

$$\limsup_k |z_k| \le A \tag{R.5}$$

with probability 1.

Proof.

Read Equation R.1 as a moving average: the new value z_{k+1} is a weighted sum of the old value z_k, with weight $1-\beta_k$, and the fresh input $c_k + \xi_{k+1}$, with weight β_k. The proof has three steps. Step 1 shows that, in the absence of noise, the condition $\sum_k \beta_k = \infty$ removes the effect of earlier values. Step 2 uses the martingale-difference and boundedness assumptions together with $\sum_k \beta_k^2 < \infty$ to show that the accumulated weighted noise converges. Step 3 combines these two conclusions.

Throughout the proof we use, without proof, one standard property of the limit superior: it is **subadditive**. For real sequences (p_k) and (q_k),

$$\limsup_k (p_k + q_k) \le \limsup_k p_k + \limsup_k q_k,$$

whenever the right-hand side is defined. In particular, if $q_k \to 0$, then $\limsup_k (p_k + q_k) \le \limsup_k p_k$: adding a sequence that vanishes cannot raise the limiting bound.

Step 1: Without noise, the average has the same limiting bound as its inputs

Claim: suppose that (u_k) and (b_k) are sequences of real numbers satisfying

$$u_{k+1} = (1-\beta_k)u_k + \beta_k b_k \tag{R.6}$$

with $0 \le \beta_k \le 1$, $\sum_k \beta_k = \infty$, and $\limsup_k |b_k| \le B$. Then $\limsup_k |u_k| \le B$.

Proof of the claim.

Fix $\varepsilon > 0$. By the definition of the limit superior, there is an index K such that

$$|b_k| \le B + \varepsilon$$

for all $k \ge K$. Call the number $B + \varepsilon$ the **ceiling**. For $k \ge K$, Equation R.6, the triangle inequality, $0 \le \beta_k \le 1$, and $|b_k| \le B + \varepsilon$ give

$$\begin{aligned} |u_{k+1}| &\le |1 - \beta_k|\,|u_k| + |\beta_k|\,|b_k| \text{(triangle inequality)} \\ &= (1 - \beta_k)|u_k| + \beta_k|b_k| (0 \le \beta_k \le 1) \\ &\le (1 - \beta_k)|u_k| + \beta_k(B + \varepsilon). (|b_k| \le B + \varepsilon) \end{aligned}$$

Subtracting the ceiling from both sides and using $\beta_k(B + \varepsilon) - (B + \varepsilon) = -(1 - \beta_k)(B + \varepsilon)$ gives

$$|u_{k+1}| - (B + \varepsilon) \le (1 - \beta_k)(|u_k| - (B + \varepsilon)). \qquad (R.7)$$

Define

$$x_k \overset{\text{def}}{=} |u_k| - (B + \varepsilon), \qquad d_k \overset{\text{def}}{=} \max\{x_k, 0\}. \qquad (R.8)$$

Thus $x_k > 0$ when $|u_k|$ is greater than $B + \varepsilon$, $x_k < 0$ when it is smaller, and $x_k = 0$ when they are equal. The quantity d_k keeps only the excess above $B + \varepsilon$: it equals $|u_k| - (B + \varepsilon)$ when this difference is positive, and it equals zero otherwise. Directly from the definition of the maximum, for every index j,

$$d_j \ge 0, \qquad x_j \le d_j. \qquad (R.9)$$

Since $|u_k| = (B + \varepsilon) + x_k$ by the definition of x_k, the second inequality in Equation R.9 gives

$$|u_k| \le B + \varepsilon + d_k.$$

Suppose for a moment that $d_k \to 0$. By the subadditivity of the limit superior stated before the proof, adding the vanishing sequence d_k cannot raise the limiting bound, so

$$\limsup_k |u_k| \le \limsup_k (B + \varepsilon + d_k) = B + \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, this gives $\lim\sup_k |u_k| \le B$, which is the claim. It remains to prove that $d_k \to 0$. We will obtain this from the one-step inequality

$$d_{k+1} \le (1 - \beta_k)\, d_k. \qquad (R.10)$$

Once this inequality is established, repeated substitution and the condition $\sum_k \beta_k = \infty$ will force $d_k \to 0$.

In this notation, Equation R.7 is

$$x_{k+1} \le (1 - \beta_k)x_k. \qquad (R.11)$$

The step-size assumption $0 \le \beta_k \le 1$ gives $1 - \beta_k \ge 0$. There are now two cases, according to the sign of x_{k+1}. If $x_{k+1} < 0$, then $d_{k+1} = \max\{x_{k+1}, 0\} = 0$, while $(1 - \beta_k)d_k$ is a product of two nonnegative numbers by Equation R.9, so

$$d_{k+1} = 0 \le (1 - \beta_k)d_k.$$

If instead $x_{k+1} \ge 0$, then $d_{k+1} = \max\{x_{k+1}, 0\} = x_{k+1}$, and Equation R.11, followed by $x_k \le d_k$ from Equation R.9 and $1 - \beta_k \ge 0$, gives the direct chain

$$\begin{aligned} d_{k+1} &= x_{k+1} \\ &\le (1 - \beta_k)x_k \\ &\le (1 - \beta_k)d_k. \end{aligned}$$

In both cases Equation R.10 holds. For example, applying Equation R.10 twice gives

$$\begin{aligned} d_{K+2} &\le (1 - \beta_{K+1})d_{K+1} \\ &\le (1 - \beta_{K+1})(1 - \beta_K)d_K. \end{aligned}$$

Repeating the same substitution gives, for every $k > K$,

$$d_k \le d_K \prod_{\ell=K}^{k-1} (1-\beta_\ell). \tag{R.12}$$

To show that the product in Equation R.12 tends to zero, we compare it with the corresponding partial sum.

For $k \ge K$, define

$$P_k \overset{\text{def}}{=} \prod_{\ell=K}^{k-1} (1-\beta_\ell), \qquad \sigma_k \overset{\text{def}}{=} \sum_{\ell=K}^{k-1} \beta_\ell, \tag{R.13}$$

with the usual conventions that an empty product equals 1 and an empty sum equals 0, so that $P_K = 1$ and $\sigma_K = 0$. The symbol σ_k is chosen here to avoid a clash with the partial sums of the noise series introduced in Step 2 below. We will prove

$$0 \le P_k \le \frac{1}{1+\sigma_k} \tag{R.14}$$

for every $k \ge K$. The right-hand side is useful because it goes to zero when σ_k goes to infinity.

The base case $k = K$ is immediate: $P_K = 1$ and $1/(1+\sigma_K) = 1/(1+0) = 1$.

Now suppose Equation R.14 holds for some $k \ge K$. Peeling off the last factor of the product and the last term of the sum in Equation R.13 gives

$$P_{k+1} = P_k(1-\beta_k), \qquad \sigma_{k+1} = \sigma_k + \beta_k.$$

Because $P_k \ge 0$ and $1-\beta_k \ge 0$, the new product satisfies $P_{k+1} \ge 0$, and multiplying the induction hypothesis by $1-\beta_k$ gives

$$P_{k+1} = P_k(1-\beta_k) \le \frac{1-\beta_k}{1+\sigma_k}. \tag{R.15}$$

Because each β_ℓ is nonnegative, σ_k is nonnegative, so $1+\sigma_k > 0$ and $1+\sigma_k+\beta_k > 0$. Also,

$$\begin{aligned}(1-\beta_k)(1+\sigma_k+\beta_k) &= 1+\sigma_k-\beta_k\sigma_k-\beta_k^2 \\ &\le 1+\sigma_k,\end{aligned} \tag{R.16}$$

because $\beta_k\sigma_k \ge 0$ and $\beta_k^2 \ge 0$. The product $(1+\sigma_k)(1+\sigma_k+\beta_k)$ is positive. Dividing Equation R.16 by this positive product does not change the direction of the inequality and gives

$$\frac{1-\beta_k}{1+\sigma_k} \le \frac{1}{1+\sigma_k+\beta_k} = \frac{1}{1+\sigma_{k+1}}.$$

Together with Equation R.15, this proves Equation R.14 for $k+1$. Therefore the bound holds for every $k \ge K$.

The assumption $\sum_k \beta_k = \infty$ means that the partial sums of all the step sizes grow without bound. The sum σ_k omits only the finitely many step sizes before index K, whose total is a fixed finite number, so $\sigma_k \to \infty$ as well.

Since $\sigma_k \to \infty$, it follows that

$$\frac{1}{1+\sigma_k} \to 0.$$

Thus, the quantity on the right in Equation R.14 tends to zero. Therefore P_k, which is nonnegative and no larger than that quantity, also tends to zero. Equation R.13 defines P_k as the product, so

$$\prod_{\ell=K}^{k-1} (1-\beta_\ell) = P_k \to 0, \quad k \to \infty. \tag{R.17}$$

Equations R.12 and Equation R.17 give $d_k \to 0$. We said earlier that, in order to claim that $\limsup_k |u_k| \le B$, it remained to prove that $d_k \to 0$, which we just did.

This proves the claim.

Step 1 proves the following fact about the update rule in Equation R.6. If

$$\limsup_k |b_k| \le B,$$

then the resulting values u_k satisfy

$$\limsup_k |u_k| \le B.$$

In Equation R.1, the value averaged in at update $k+1$ is $c_k + \xi_{k+1}$. We know a limiting bound for c_k, but boundedness of ξ_{k+1} does not imply that $\xi_{k+1} \to 0$. For example, a bounded sequence may alternate between C and $-C$ forever. Therefore we cannot yet conclude that $c_k + \xi_{k+1}$ has the same limiting bound as c_k, so Step 1 does not yet prove the lemma.

Equation R.1 contains the weighted random term $\beta_k \xi_{k+1}$. Step 2 proves that the partial sums

$$\sum_{k=0}^{n-1} \beta_k \, \xi_{k+1}$$

converge as $n \to \infty$. Step 3 uses this convergence to transform Equation R.1 into an update rule of the form studied in Step 1, with one additional input that tends to zero. Step 1 can then be used to complete the proof.

Step 2: The accumulated noise converges

Let C be the bound on the noise, so $|\xi_{k+1}| \le C$ for all k, and consider the accumulated weighted noise

$$S_n \overset{\text{def}}{=} \sum_{k=0}^{n-1} \beta_k \, \xi_{k+1}. \tag{R.18}$$

To prove that S_n converges, we use the following classical result without proof.

Doob's martingale convergence theorem (the special case used here). Suppose that

$$\mathbb{E}[\beta_k \xi_{k+1} \mid H_k] = 0$$

for every k, and that

$$\sum_k \mathbb{E}\left[(\beta_k \xi_{k+1})^2 \mid H_k\right] < \infty.$$

Then the series $\sum_k \beta_k \, \xi_{k+1}$ converges to a finite limit with probability 1.

We now verify these two conditions. First, because β_k is fixed once H_k is fixed, Equation R.2 gives

$$\begin{aligned} \mathbb{E}[\beta_k \xi_{k+1} \mid H_k] &= \beta_k \mathbb{E}[\xi_{k+1} \mid H_k] \\ &= 0. \end{aligned} \tag{R.19}$$

Second, the bound $|\xi_{k+1}| \le C$ implies

$$\xi_{k+1}^2 \le C^2.$$

Because $\beta_k^2 \ge 0$, multiplying this inequality by β_k^2 gives

$$(\beta_k \xi_{k+1})^2 = \beta_k^2 \xi_{k+1}^2 \le C^2 \beta_k^2.$$

Once H_k is fixed, the quantity $C^2 \beta_k^2$ is fixed. Therefore taking the conditional expectation preserves this upper bound:

$$\mathbb{E}[(\beta_k \xi_{k+1})^2 \mid H_k] \le C^2 \beta_k^2. \tag{R.20}$$

Summing Equation R.20 over k and using Equation R.3 gives

$$\sum_k \mathbb{E}\left[(\beta_k \xi_{k+1})^2 \mid H_k\right] \le C^2 \sum_k \beta_k^2 < \infty. \tag{R.21}$$

Equation R.19 verifies the first condition in Doob's theorem, and Equation R.21 verifies the second. Therefore Doob's theorem shows that the series

$$\sum_k \beta_k \xi_{k+1}$$

converges to a finite limit with probability 1. Its n^{th} partial sum is S_n by Equation R.18. Hence there is a finite random limit S_∞ such that

$$S_n \to S_\infty, \quad n \to \infty$$

with probability 1. This proves that the accumulated weighted noise converges.

Step 3: Absorb the noise into the input

Step 2 shows that, with probability 1, the numerical sequence $S_0, S_1, S_2, \ldots$ obtained from the observed noise values converges to a finite limit. Consider any possible sequence of observed noise values for which this convergence occurs, and keep those values fixed for the rest of the proof. For this fixed sequence, write the limit as S_∞ and define the **tail**

$$\zeta_k \stackrel{\text{def}}{=} S_\infty - S_k = \sum_{j=k}^{\infty} \beta_j \ \xi_{j+1}, \tag{R.22}$$

the total weighted noise the process has not yet received. Since $S_k \to S_\infty$, Equation R.22 gives

$$\zeta_k = S_\infty - S_k \to 0, \quad k \to \infty. \tag{R.23}$$

Equation R.18 gives

$$S_{k+1} = S_k + \beta_k \xi_{k+1}.$$

Substituting this equality into Equation R.22 gives

$$\begin{aligned} \zeta_{k+1} &= S_\infty - S_{k+1} \\ &= S_\infty - S_k - \beta_k \xi_{k+1} \\ &= \zeta_k - \beta_k \xi_{k+1}. \end{aligned} \tag{R.24}$$

Define a new sequence by

$$\tilde{z}_k \stackrel{\text{def}}{=} z_k + \zeta_k. \tag{R.25}$$

Thus $\tilde{z}_k$ is the value z_k after update k, plus the tail ζ_k of the accumulated weighted noise defined in Equation R.22. Since

$$(1 - \beta_k) + \beta_k = 1,$$

we have the algebraic identity

$$\zeta_k = (1 - \beta_k)\zeta_k + \beta_k \zeta_k. \tag{R.26}$$

Equations R.1[†], Equation R.24[‡], Equation R.26[§], and the definition of $\tilde{z}_k$ in Equation R.25[¶] now give

$$\begin{aligned} \tilde{z}_{k+1} &= z_{k+1} + \zeta_{k+1} \\ &= (1 - \beta_k) z_k + \beta_k c_k + \beta_k \xi_{k+1} + \zeta_k - \beta_k \xi_{k+1} \qquad \text{(by † and ‡)} \\ &= (1 - \beta_k) z_k + \beta_k c_k + \beta_k \xi_{k+1} + \zeta_k - \beta_k \xi_{k+1} \qquad \text{(noise terms cancel)} \\ &= (1 - \beta_k) z_k + \beta_k c_k + \zeta_k \\ &= (1 - \beta_k) z_k + \beta_k c_k + (1 - \beta_k)\zeta_k + \beta_k \zeta_k \qquad \text{(by §)} \\ &= (1 - \beta_k)(z_k + \zeta_k) + \beta_k (c_k + \zeta_k) \\ &= (1 - \beta_k)\tilde{z}_k + \beta_k (c_k + \zeta_k). \qquad \text{(by ¶)} \end{aligned} \tag{R.27}$$

Equation R.27 has exactly the form of the update rule in Equation R.6, with

$$b_k = c_k + \zeta_k. \tag{R.28}$$

Fix $\varepsilon > 0$. Equation R.4 implies that eventually

$$|c_k| \le A + \frac{\varepsilon}{2},$$

and Equation R.23 implies that eventually

$$|\zeta_k| \le \frac{\varepsilon}{2}.$$

For all sufficiently large k, the triangle inequality therefore gives

$$|b_k| = |c_k + \zeta_k| \le |c_k| + |\zeta_k| \le A + \varepsilon.$$

Hence

$$\limsup_k |b_k| \le A.$$

Step 1, applied to Equation R.27 with $B = A$, gives

$$\limsup_k |\tilde{z}_k| \le A. \tag{R.29}$$

The definition $\tilde{z}_k = z_k + \zeta_k$ gives

$$z_k = \tilde{z}_k - \zeta_k.$$

The triangle inequality gives, for every k,

$$|z_k| = |\tilde{z}_k - \zeta_k| \le |\tilde{z}_k| + |\zeta_k|.$$

Taking the limit superior, using its subadditivity, and then applying Equations R.29 and Equation R.23 gives

$$\begin{aligned} \limsup_k |z_k| &\le \limsup_k |\tilde{z}_k| + \limsup_k |\zeta_k| \\ &\le A + 0 \\ &= A. \end{aligned}$$

Step 2 shows that, with probability 1, the sequence S_n converges to a finite limit S_∞. Whenever this convergence occurs, the tail $\zeta_k = S_\infty - S_k$ is defined and satisfies $\zeta_k \to 0$. The calculations in Step 3 then show, for those same observed noise values, that

$$\limsup_k |z_k| \le A.$$

Thus the desired bound can fail only for observed noise values for which S_n does not converge. Step 2 shows that the probability of this failure is 0. Therefore,

$$\limsup_k |z_k| \le A$$

with probability 1. This proves the lemma.

Remark (pathwise form). Only Step 2 of the proof is probabilistic. Once a noise realization for which the series $\sum_k \beta_k \xi_{k+1}$ converges has been fixed, Steps 1 and 3 are ordinary deterministic arguments about the resulting sequences of numbers. The lemma therefore holds in the following stronger form: on the probability-one event where the accumulated noise converges, every realization whose input sequence satisfies $\lim \sup_k |c_k| \le A$—where the number A is now allowed to depend on the realization—also satisfies $\lim \sup_k |z_k| \le A$ for that same realization. The asynchronous stochastic-approximation theorem uses the lemma in this pathwise form, with a bound that depends on the realized error sequence.